Homework 3

1. **Sum of Poisson.** Let $\mathbb{X}$ be the Poisson distribution with mean m/n. Let $\mathbb{S}_n := \mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \cdots + \mathbb{X}^{(n)}$, where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \ldots, \mathbb{X}^{(n)}$ are all independent and identical to $\mathbb{X}$. Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

Solution.

2. **Sum of an Interesting Random Variable.** (20 points) Let $\mathbb{X}$ be the random variable over natural numbers $\{1, 2, 3, \dots\}$ such that, for any natural number i, we have

$$\mathbb{P}\left[\mathbb{X} = i\right] = 2^{-i}$$

Let $\mathbb{S}_n = \mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \cdots + \mathbb{X}^{(n)}$, where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(n)}$ are independent and identical to $\mathbb{X}$.

- (5 points) What is $\mathbb{E}\left[\mathbb{S}_n\right]$?
- (15 points) Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

Solution.

3. **Coin-tossing: Word Problem.** (20 points) Suppose you have access to a coin that outputs heads with probability 1/2 and outputs tails with probability 1/2. Let $\mathbb{S}_n$ represent the number of coin tosses needed to see exactly n heads.

   - (5 points) What is $\mathbb{E}[\mathbb{S}_n]$?
   - (15 points) Upper-bound the following probability

   $$\mathbb{E}\left[\mathbb{S}_n - \mathbb{E}[\mathbb{S}_n] \geqslant E\right]$$

   Solution.

4. **Empty Bins in the Poisson Model.** (20 points) Let $\mathbb{X}$ represent the Poisson distribution with mean m/n. Let $\mathbb{Y}$ be the indicator variable $\mathbf{1}_{\{\mathbb{X}=0\}}$. That is, $\mathbb{Y}$ is the random variable that is 1 if and only if the load is 0.

   Let $\mathbb{S}_n = \mathbb{Y}^{(1)} + \mathbb{Y}^{(2)} + \ldots + \mathbb{Y}^{(n)}$, where $\mathbb{Y}^{(1)}, \mathbb{Y}^{(2)}, \ldots, \mathbb{Y}^{(n)}$ are independent and identical to $\mathbb{Y}$.

   - (5 points) What is $\mathbb{E}\left[\mathbb{S}_n\right]$?
   - (15 points) Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

   Solution.

5. **Random Walk in 2-D.** (20 points) Suppose an insect starts at $(0,0)$ at time $t=0$. At time t, its position is described by $(\mathbb{X}(t), \mathbb{Y}(t))$. At the next time step $t+1$, the insect uniformly at random moves to (a) $(\mathbb{X}(t)+1, \mathbb{Y}(t))$, $(\mathbb{X}(t)-1, \mathbb{Y}(t))$, $(\mathbb{X}(t), \mathbb{Y}(t)+1)$, or $(\mathbb{X}(t), \mathbb{Y}(t)-1)$.

   State (5 points) and prove (15 points) a theorem that bounds how far from the origin the insect is at time $t=n$.

   Solution.

Collaborators :